

A Cayley graph for $F_2 \times F_2$ which is not minimally almost convex.

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Abstract

We give an example of a Cayley graph Γ for the group $F_2 \times F_2$ which is not minimally almost convex (MAC). On the other hand, the standard Cayley graph for $F_2 \times F_2$ does satisfy the falsification by fellow traveler property (FFTP). As a result, we show that any Cayley graph property K which satisfies $\text{FFTP} \Rightarrow K \Rightarrow \text{MAC}$ is dependent on the generating set. This includes the well known properties FFTP and almost convexity, which were already known to depend on the generating set. This also shows the new results that Poenaru's condition $P(2)$ and the basepoint loop shortening property both depend on the generating set. We also show that the Cayley graph Γ does not have the loop shortening property, so this property also depends on the generating set.

1 Introduction

In this note, we consider two different presentations for the group $F_2 \times F_2$, whose Cayley graphs exhibit quite different geometry. First we consider the standard presentation

$$(G, S_1) = \langle a, b, c, d \mid ac = ca, bc = cb, ad = da, bd = db \rangle.$$

Note that this is a right angled Artin group with the standard presentation, so by theorem 3.1 in [5], the pair (G, S_1) satisfies the falsification by fellow traveller property. The other presentation that we will consider is

$$(G, S_2) = \langle x, y, c, d \mid xc = cx, yc = cy, xcd = dxc, ycd = dyc \rangle.$$

These presentations can be seen to define the same group by the Tietze transformations below:

$$\begin{aligned} &\langle a, b, c, d \mid ac = ca, bc = cb, ad = da, bd = db \rangle \\ &= \langle x, y, a, b, c, d \mid x = ac^{-1}, y = bc^{-1}, ac = ca, bc = cb, ad = da, bd = db \rangle \\ &= \langle x, y, a, b, c, d \mid a = xc, b = yc, xcc = cxc, ycc = cy c, xcd = dxc, ycd = dyc \rangle \\ &= \langle x, y, c, d \mid xcc = cxc, ycc = cy c, xcd = dxc, ycd = dyc \rangle \\ &= \langle x, y, c, d \mid xc = cx, yc = cy, xcd = dxc, ycd = dyc \rangle. \end{aligned}$$

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Our main theorem is that the pair (G, S_2) is not minimally almost convex. As a corollary, the following properties all depend on the generating set:

- The falsification by fellow traveler property
- The basepoint loop shortening property
- Every non-trivial almost convexity condition. This includes almost convexity, Poenaru's condition $P(2)$, minimal almost convexity (MAC) and the slightly stronger condition M'AC.

In the final section we show that (G, S_2) does not satisfy the loop shortening property, so this property also depends on the generating set.

There are many known examples of groups which satisfy FFTP with respect to one generating set but not others. For example, Neumann and Shapiro gave an example in [6] when they introduced the property, and Elder gave another example in [3]. The only previous example for almost convexity was given by Theil [8]. Elder and Hermiller used solvable Baumslag-Solitar groups to show that minimal almost convexity is not a quasi-isometry invariant, nor is M'AC [4], so it is not surprising that these both depend on the generating set. It was previously unknown, however, whether Poenaru's condition $P(2)$ or either of the loop shortening properties were dependent on the generating set.

2 almost convexity conditions

In the following definition, let $Sph(r)$ denote the sphere of radius r around the identity in the Cayley graph, and let $\overline{B(r)}$ denote the ball of radius r .

Definition 2.1. Let G be a group and let S be a finite generating set for G . Let $r_0 \in \mathbb{Z}_{>0}$ and let $f : \mathbb{Z}_{>0} \rightarrow \mathbb{Z}_{>0}$ be a function. The pair (G, S) is said to satisfy the almost convexity condition AC_{f,r_0} if the following holds: If $r \in \mathbb{Z}_{>r_0}$ and $u, v \in Sph(r)$ satisfy $d(u, v) \leq 2$, then there is some path p between u and v such that p is contained in $\overline{B(r)}$ and p has length at most $f(r)$.

Using this definition, almost convexity (AC) imposes the strongest possible restriction on f , namely that f is a constant function. This property was introduced by Cannon in [1], and remains the most widely used almost convexity condition. A pair (G, S) is said to satisfy Poenaru's condition $P(2)$ [7] if it satisfies an almost convexity condition AC_{f,r_0} for some sub-linear function f .

Note that the points u and v in the definition above are joined via the origin by a path of length $2r$, so every finitely generated group trivially satisfies any almost convexity condition where $f(r) \geq 2r$. Hence, minimal almost convexity (MAC) requires the weakest possible non-trivial almost convexity condition: that $f(r) = 2r - 1$. The slightly stronger condition M'AC requires that $f(r) = 2r - 2$. Due to the successive strengthening of these conditions, we have the implication chain

$$AC \Rightarrow P(2) \Rightarrow M'AC \Rightarrow MAC.$$

The last two conditions M'AC and MAC are discussed in greater detail in [4], where Elder and Hermiller show that M'AC does not imply $P(2)$. It is still an open question, however, whether the properties M'AC and MAC are equivalent.

3 The pair (G, S_2) does not satisfy MAC

The main theorem in this section is that the group (G, S_2) with presentation

$$\langle a, b, c, t \mid ac = ca, bc = cb, act = tac, bct = tbc \rangle$$

does not satisfy MAC.

Consider the group $P = F_2 \times C_\infty = \langle a, b, c \mid ac = ca, bc = cb \rangle$ and let H be the subgroup generated by ac and bc . Then G is the HNN extension $P *_\phi$ where $\phi : H \rightarrow H$ is the identity.

Lemma 3.1. *H is the subgroup of G consisting of all elements wc^k , where w is a word in $\{a, b, a^{-1}, b^{-1}\}^*$ and k is the sum of the exponents in w .*

Proof. Let w be a word in $\{a, b, a^{-1}, b^{-1}\}^*$ and let k be the sum of the exponents in w . We will show that $wc^k \in H$. Let $w = s_1^{p_1} s_2^{p_2} \dots s_n^{p_n}$, where each $s_i \in \{a, b\}$ and each $p_i \in \{-1, 1\}$. Then

$$wc^k = wc^{p_1 + \dots + p_n} = (s_1 c)^{p_1} (s_2 c)^{p_2} \dots (s_n c)^{p_n} \in H.$$

Now let $h \in H$. We will show that $h = wc^k$ for some word $w \in \{a, b, a^{-1}, b^{-1}\}^*$, where k is the sum of the exponents in w . Since $h \in H = \langle ac, bc \rangle$, we can write

$$h = (s_1 c)^{p_1} (s_2 c)^{p_2} \dots (s_n c)^{p_n},$$

where each $s_i \in \{a, b\}$ and each $p_i \in \{-1, 1\}$. Then we have

$$h = s_1^{p_1} s_2^{p_2} \dots s_n^{p_n} c^{p_1 + p_2 + \dots + p_n} = wc^k,$$

where $w = s_1^{p_1} s_2^{p_2} \dots s_n^{p_n}$ and k is the sum of the exponents in w . □

Theorem 3.2. *The pair (G, S_2) does not satisfy MAC.*

Proof. Consider the elements $a^n b^{-n}$ and $ta^n b^{-n+1}$ in G . Then

$$|a^n b^{-n}| = 2n = |ta^n b^{-n+1}|$$

and

$$d(a^n b^{-n}, ta^n b^{-n+1}) = |b^n a^{-n} ta^n b^{-n+1}| = |tb^n a^{-n} a^n b^{-n+1}| = |tb| = 2.$$

So we just need to prove that if p is a path in $B(2n)$ between $a^n b^{-n}$ and $ta^n b^{-n+1}$, then p has length at least $4n$.

Let p be a path in $B(2n)$ between $a^n b^{-n}$ and $ta^n b^{-n+1}$.

Then let h and ht be adjacent vertices in p , such that h is in the same sheet of the HNN extension as 1 and ht is in the same sheet as t . Then $h \in H$. Moreover, since p is contained in $B(2n)$, the length $|h| + 1 = |ht| \leq 2n$, so $|h| \leq 2n - 1$. Let $h = wc^k$ where w is a word in $\{a, b, a^{-1}, b^{-1}\}^*$ and k is the sum of the exponents in w . Then

$$d(a^n b^{-n}, h) = |b^n a^{-n} h| = |b^n a^{-n} wc^k| = |b^n a^{-n} w| + |k|.$$

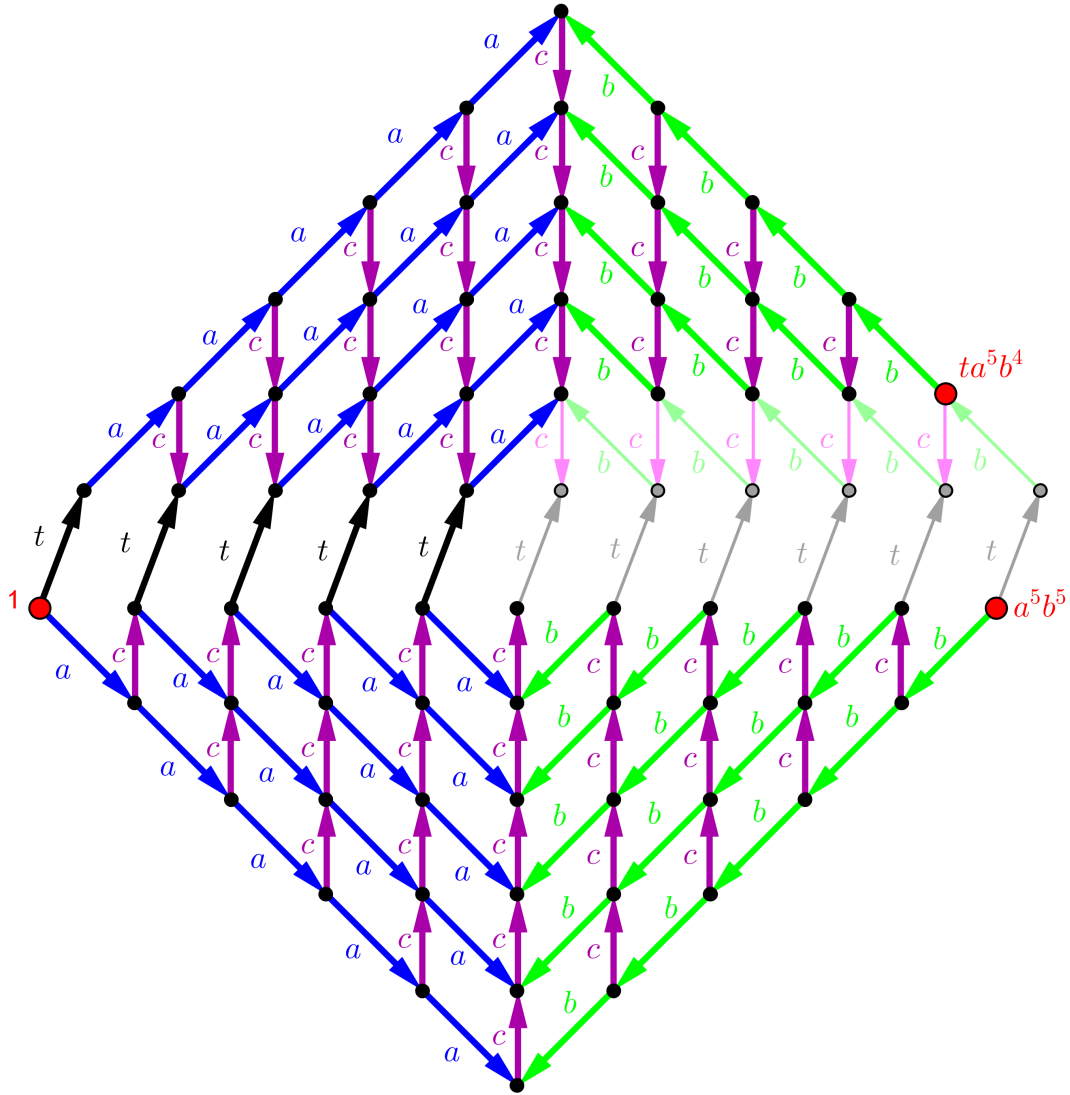


Figure 1: Part of the Cayley graph $\Gamma(G, S_2)$. The bold vertices and edges are those within the ball of radius 10.

Let $x \in \mathbb{Z}_{\geq 0}$ be maximal such that the word w splits as $w = a^x w_1$. In other words, x is the number of a 's at the start of w . Then the sum of the exponents in w_1 is $k - x$, so $|w_1| \geq |k - x|$. Therefore,

$$2n - 1 \geq |h| = |w| + |k| = x + |w_1| + |k| \geq x + |k - x| + |k| \geq 2x,$$

so $x < n$. Therefore, $b^n a^{x-n} w_1$ is freely reduced, so

$$|b^n a^{-n} a^x w_1| = |b^n a^{x-n} w_1| = 2n - x + |w_1|.$$

Therefore,

$$d(a^n b^{-n}, h) = |b^n a^{-n} w| + |k| = 2n - x + |w_1| + |k| \geq 2n - x + x = 2n.$$

Hence, the path p satisfies

$$|p| \geq d(a^n b^{-n}, h) + 1 + d(ta^n b^{1-n}, ht) \geq d(a^n b^{-n}, h) + d(ta^n b^{-n}, ht) = 2d(a^n b^{-n}, h) \geq 4n,$$

as required. So (G, S_2) does not satisfy MAC. \square

4 loop shortening properties

In [2] Elder introduced the loop shortening and basepoint loop shortening properties as a natural generalisation of the falsification by fellow traveller property. Where FFTP gives a simple way to check if a word is a geodesic, each of the loop shortening properties gives a somewhat simple way to check if a word represents the identity in the group.

Definition 4.1. Let G be a group with finite generating set S . (G, S) has the (synchronous) loop shortening property (LSP) if there is a constant k such that for any loop v_0, v_1, \dots, v_n in $\Gamma(G, S)$ with $n \geq 1$, there is a shorter loop u_0, u_1, \dots, u_m such that $d(u_j, v_j) < k$ for each $j \leq m$, and $d(u_m, v_j) < k$ for $m \leq j \leq n$. In other words, the paths (synchronously) k -fellow travel.

Definition 4.2. Let G be a group with finite generating set S . (G, S) has the (synchronous) basepoint loop shortening property (BLSP) if there is a constant k such that for any loop v_0, v_1, \dots, v_n in $\Gamma(G, S)$ with $n \geq 1$, there is a shorter loop $(v_0 = u_0), u_1, \dots, u_m$ such that $d(u_j, v_j) < k$ for each $j \leq m$, and $d(u_m, v_j) < k$ for $m \leq j \leq n$. In other words, the paths (synchronously) k -fellow travel.

Note that the only difference between these two properties is that for the basepoint loop shortening property, the initial loop is around a basepoint which the shorter loop has to pass through, whereas for the loop shortening property no such restriction is imposed. Hence, it is clear that

$$\text{BLSP} \Rightarrow \text{LSP}.$$

Elder also showed that the basepoint loop shortening property is strictly stronger than almost convexity and strictly weaker than the falsification by fellow traveller property (FFTP), so we have the long implication chain

$$\text{FFTP} \Rightarrow \text{BLSP} \Rightarrow \text{AC} \Rightarrow \text{P}(2) \Rightarrow \text{M}'\text{AC} \Rightarrow \text{MAC}.$$

Elder asked two questions about the two loop shortening properties. The first is whether they are equivalent, and this remains an open problem. The second is whether either or both of these properties depend on the generating set. We have already shown that the group $G = F_2 \times F_2$ satisfies FFTP with respect to one generating set, but fails MAC with another, which implies that BLSP depends on the generating set. Our final theorem settles the other half of this question, namely that the loop shortening property also depends on the generating set.

Theorem 4.3. *The group (G, S_2) with presentation*

$$\langle a, b, c, t \mid ac = ca, bc = cb, act = tac, bct = tbc \rangle$$

does not satisfy the loop shortening property.

Proof. Let $k \in \mathbb{Z}_{>0}$. We will show that there is a loop l in $\Gamma(G, S_2)$ such that there is no shorter loop l' in $\Gamma(G, S_2)$ which k -fellow travels with l . Hence this will show that $\Gamma(G, S_2)$ does not satisfy the loop shortening property. Let l be the loop given by the word

$$w = a^{2k}b^{-4k}a^{2k}ta^{-2k}b^{4k}a^{-2k}t^{-1}.$$

We can easily check algebraically that $\bar{w} = 1$, so this is indeed a loop. Now let l' be a loop which k -fellow travels with l . Then we just need to show that the length of l' is at least the length of l , which is $16k + 2$. Since the four vertices

$$u_1 = a^k, \quad u_2 = a^{2k}b^{-4k}a^k, \quad u_3 = ta^{2k}b^{-4k}a^k \quad \text{and} \quad u_4 = ta^k,$$

appear in l in that order, there must be vertices v_1, v_2, v_3 and v_4 appearing in l' in that order which satisfy $d(u_i, v_i) \leq k$ for each $i \in \{1, 2, 3, 4\}$. Hence it suffices to prove that

$$d(v_1, v_2) + d(v_2, v_3) + d(v_3, v_4) + d(v_4, v_1) \geq 16k + 2.$$

For each i , let w_i be a word of minimal length from u_i to v_i . So $|w_i| \leq k$ and $\bar{w}_i = u_i^{-1}v_i$. For each $i \in \{1, 2, 3, 4\}$, let x_i, y_i, z_i be the sums of the powers of a, b and c , respectively in the word w_i . Then we will show the following four inequalities, from which the desired result follows:

$$d(v_1, v_2) \geq 6k - x_1 + x_2 + |y_1| + |y_2| + |z_1 - z_2|,$$

$$d(v_2, v_3) \geq 2k + 1 - x_2 - x_3 - y_2 - y_3 + z_2 + z_3,$$

$$d(v_3, v_4) \geq 6k - x_4 + x_3 + |y_4| + |y_3| + |z_4 - z_3|,$$

$$d(v_4, v_1) \geq 2k + 1 + x_1 + x_4 + y_1 + y_4 - z_1 - z_4.$$

Let w be a word of minimal length from v_1 to v_2 . So

$$\bar{w} = \bar{w}_1^{-1}u_1^{-1}u_2\bar{w}_2 = \overline{w_1^{-1}a^kb^{-4k}a^kw_2}.$$

Consider the quotient map $f : G \rightarrow G$ defined by $f(a) = a, f(b) = b, f(c) = c$ and $f(t) = 1$. Note that this is well defined since $f(ac) = f(ca), f(bc) = f(cb), f(act) = f(tac)$ and $f(bct) = f(tbc)$. Since c commutes with a and b , we can write $f(\bar{w}_1) = c^{z_1}\bar{r}_1$, where

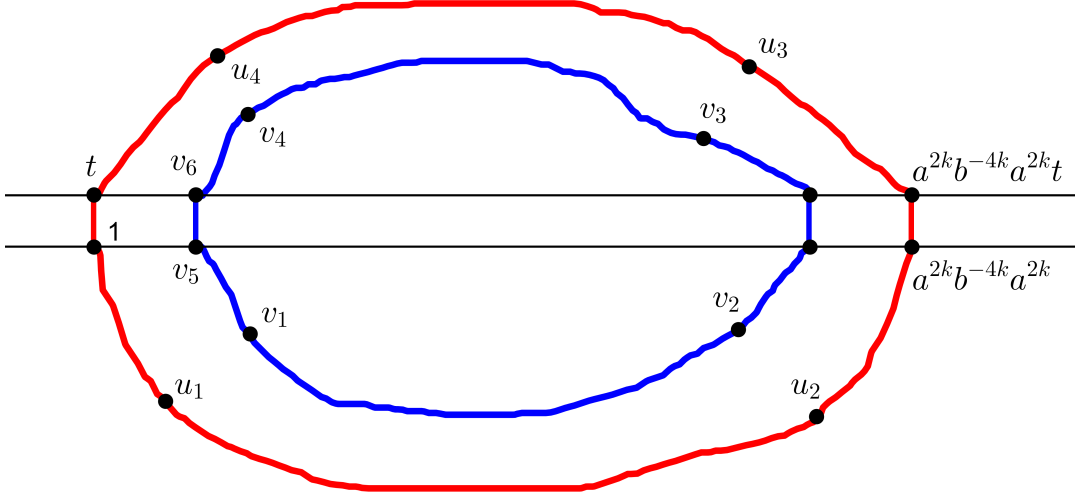


Figure 2: The loop l in red and a k -fellow travelling loop l' in blue.

r_1 is a reduced word over the alphabet $\{a, b, a^{-1}, b^{-1}\}$. Then x_1 and y_1 are the sums of the powers of a and b , respectively, in r_1 and

$$k \geq |w_1| \geq |r_1|.$$

Similarly we can write $f(\overline{w_2}) = c^{z_2}\overline{r_2}$. Now,

$$d(v_1, v_2) = |w| \geq d(1, f(\overline{w})) = d(1, f(\overline{w_1})^{-1}a^kb^{-4k}a^kf(\overline{w_2})) = d(1, c^{z_2-z_1}\overline{r_1}^{-1}a^kb^{-4k}a^k\overline{r_2}).$$

Since r_1 and r_2 each have length at most k , they can only cancel with letters in a^k on either side of the last expression, and not the b^{-4k} term in the middle. In other words, if s_1 is a reduced word for $\overline{r_1}^{-1}a^k$ and s_2 is a reduced word for $a^k\overline{r_2}$, then $c^{z_2-z_1}s_1b^{-4k}s_2$ is a reduced word for $c^{z_2-z_1}\overline{r_1}^{-1}a^kb^{-4k}a^k\overline{r_2}$. Moreover, the sums of the powers of a in s_1 and s_2 are $k - x_1$ and $k + x_2$ respectively, and the sums of the powers of b in s_1 and s_2 are $-y_1$ and y_2 respectively. Hence, $|s_1| \geq k - x_1 + |y_1|$ and $|s_2| \geq k + x_2 + |y_2|$. Therefore,

$$\begin{aligned} d(v_1, v_2) &\geq d(1, c^{z_2-z_1}\overline{r_1}^{-1}a^kb^{-4k}a^k\overline{r_2}) \\ &= |c^{z_2-z_1}s_1b^{-4k}s_2| \\ &= 4k + |z_2 - z_1| + |s_1| + |s_2| \\ &\geq 4k + |z_2 - z_1| + k - x_1 + |y_1| + k + x_2 + |y_2| \\ &= 6k - x_1 + x_2 + |y_1| + |y_2| + |z_1 - z_2|. \end{aligned}$$

Similarly,

$$d(v_3, v_4) \geq 6k - x_4 + x_3 + |y_4| + |y_3| + |z_4 - z_3|.$$

Now let r be a word of minimal length between v_2 and v_3 . Then $w_2rw_3^{-1}$ forms a path p from u_2 to u_3 . Since u_2 is in the sheet of the HNN extension containing 1 and u_3 is in the sheet containing t , there must be vertices v_5 and v_6 which are adjacent in p and such that v_5 is in the sheet containing 1 and v_6 is in the sheet containing t . So $v_6 = v_5t$ and $v_5 \in H$,

the subgroup generated by ac and bc . Let $C_1 = \langle e \rangle$ be the one generator cyclic group, and let $h : G \rightarrow C_1$ be the group homomorphism defined by $h(a) = h(b) = e$, $h(c) = e^{-1}$ and $h(t) = 1$. Then H is in the kernel of h , so $h(v_5) = h(v_6) = 1$. Since $h(u_2) = h(u_3) = e^{-k}$, the distances $d(u_2, v_5)$ and $d(u_3, v_6)$ are both at least k . But the sections of p joining u_2 and v_2 and v_3 to u_3 both have length at most k . Hence v_5 and v_6 lie on the section of p between v_2 and v_3 . Hence, the distance

$$d(v_2, v_3) = d(v_2, v_5) + 1 + d(v_3, v_6).$$

Now $h(v_2) = h(u_2)h(\overline{w_2}) = e^{-k}e^{x_2}e^{y_2}e^{-z_2} = e^{-k+x_2+y_2-z_2}$. Therefore,

$$d(v_2, v_5) \geq k - x_2 - y_2 + z_2.$$

Similarly,

$$d(v_3, v_6) \geq k - x_3 - y_3 + z_3.$$

Putting these together gives

$$d(v_2, v_3) = d(v_2, v_5) + 1 + d(v_3, v_6) \geq 2k + 1 - x_2 - x_3 - y_2 - y_3 + z_2 + z_3.$$

In exactly the same way, we deduce the final inequality,

$$d(v_4, v_1) \geq 2k + 1 + x_1 + x_4 + y_1 + y_4 - z_1 - z_4.$$

So we now have lower bounds for all four of the distances $d(v_1, v_2)$, $d(v_2, v_3)$, $d(v_3, v_4)$, $d(v_4, v_1)$. Therefore, the length of the loop l' is at least

$$\begin{aligned} & d(v_1, v_2) + d(v_2, v_3) + d(v_3, v_4) + d(v_4, v_1) \\ & \geq 6k - x_1 + x_2 + |y_1| + |y_2| + |z_1 - z_2| + 2k + 1 - x_2 - x_3 - y_2 - y_3 + z_2 + z_3 \\ & \quad + 6k - x_4 + x_3 + |y_4| + |y_3| + |z_4 - z_3| + 1 + x_1 + x_4 + y_1 + y_4 - z_1 - z_4 \\ & = 16k + 2 + |y_1| + y_1 + |y_2| - y_2 + |y_3| - y_3 + |y_4| + y_4 \\ & \quad + |z_1 - z_2| - (z_1 - z_2) + |z_4 - z_3| - (z_4 - z_3) \\ & \geq 16k + 2, \end{aligned}$$

which is the same as the length of l . Hence the pair (G, S_2) does not enjoy the loop shortening property. \square

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